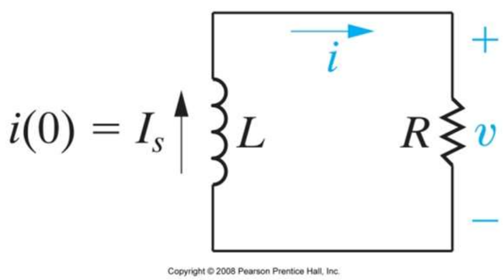
Response of First-Order RL/RC Circuits

Objectives:

- Natural response of RL/RC circuits
- Step response of RL/RC circuits
- Sequential switching in first-order circuits

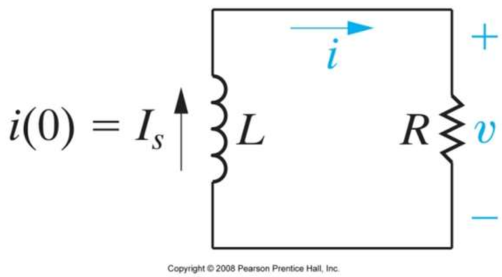
RL Natural Response



The problem: Given an initial current $i(0)$ through L at $t = 0$, find $i(t)$ for $t \geq 0$.

Note: this is the "natural" response of this circuit because for $t \geq 0$ there is no independent source in the circuit – the response is due only to the "nature" of the components and their interconnection.

RL Natural Response



$$\text{KVL}: \qquad L\frac{di(t)}{dt} + Ri(t) = 0$$

This equation is a

- First-order (the highest derivative is a first derivative)
- Homogeneous (the right-hand side is 0)
- Ordinary differential equation
- With constant coefficients

The solution to the differential equation, repeated below, must be a function whose first-derivative has the same form as the function. Which one of these functions exhibits this behavior?

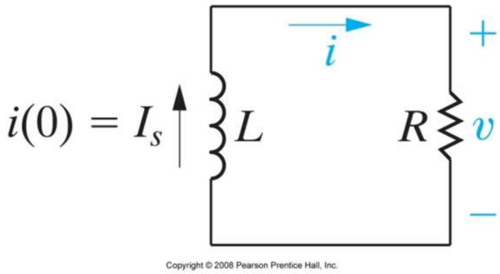
$$\text{KVL}: \qquad L\frac{di(t)}{dt} + Ri(t) = 0$$

A. $K \sin \omega t$

B. Ke^{-at}

C. Kt^2

RL Natural Response



KVL: $$L\frac{di(t)}{dt} + Ri(t) = 0$$

Let $i(t) = e^{-at}$ and $\dfrac{di(t)}{dt} = -ae^{-at}$

Substituting, $L(-ae^{-at}) + \mathrm{R}e^{-at} = 0$

Rearranging, $(R - aL)e^{-at} = 0$ so $R - aL = 0$

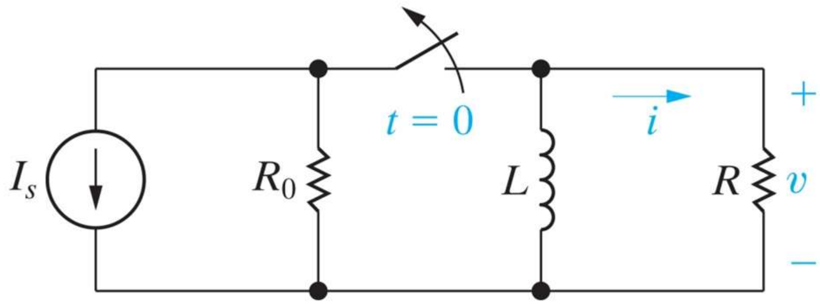
Solving, $a = R/L$

But $i(0) = I_s$ so $i(t) = I_s e^{-(R/L)t}, \quad t \geq 0$

Check: $L(-R/L)I_s e^{-(R/L)t} + RI_s e^{-(R/L)t}$

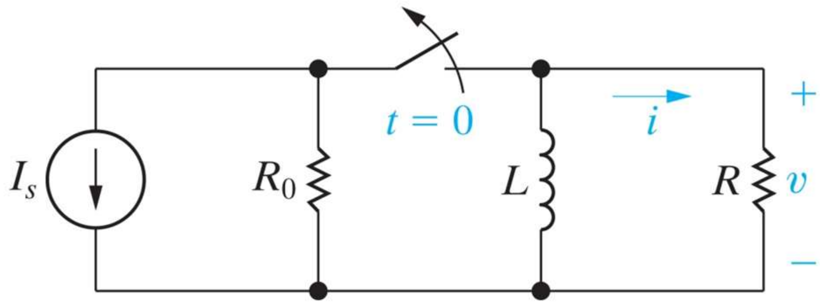
$$= -RI_s e^{-(R/L)t} + RI_s e^{-(R/L)t} = 0$$

$$i(0) = I_s e^{-(R/L)(0)} = I_s$$



Here is one circuit that can be used to establish the initial current through the inductor. For $t < 0$, the inductor can be replaced by

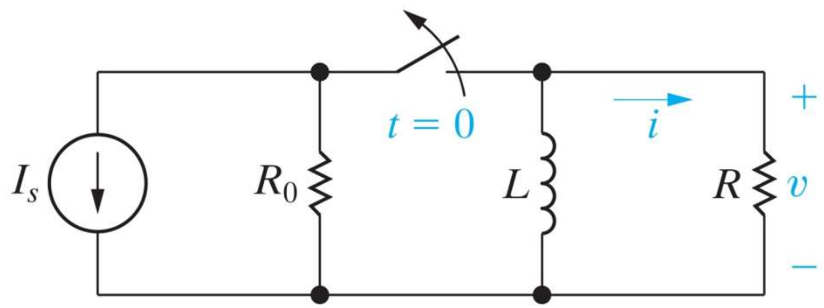
- ✗ A. An open circuit
- ✓ B. A short circuit
- ✗ C. A resistor



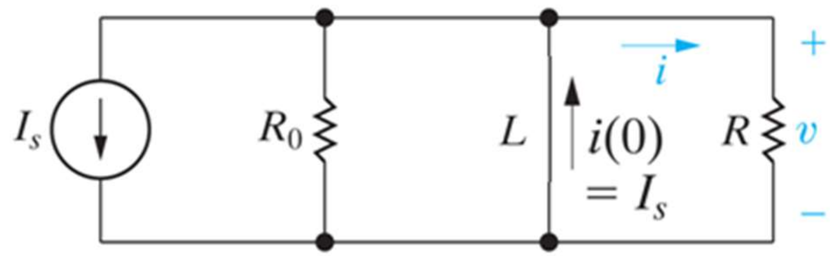
To evaluate the current in the inductor for t < 0, close the switch and replace the inductor with a short circuit. The current from bottom to top in the short circuit is

- A. $-I_S$
- B. $RI_S/(R + R_0)$
- C. I_S

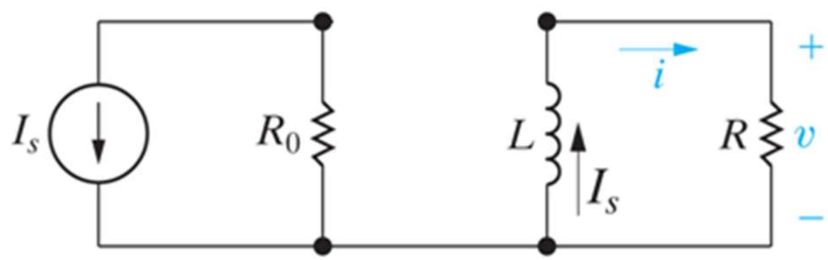
RL Natural Response



$t < 0:$

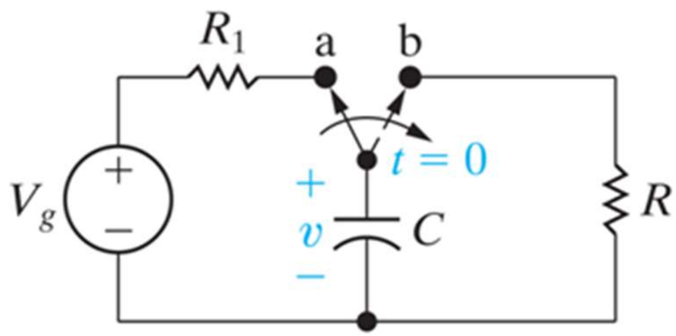


$t \geq 0:$

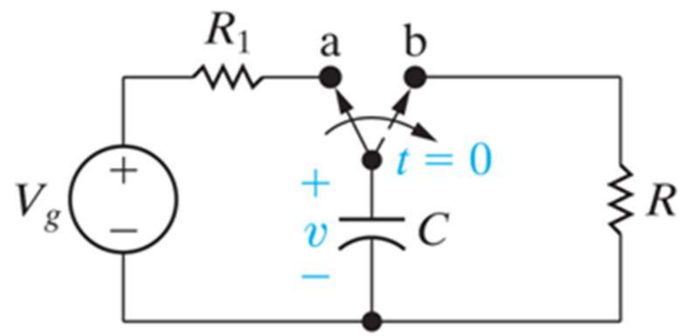


$$i(t) = I_s e^{-(R/L)t}, \qquad t \geq 0$$

RC Natural Response

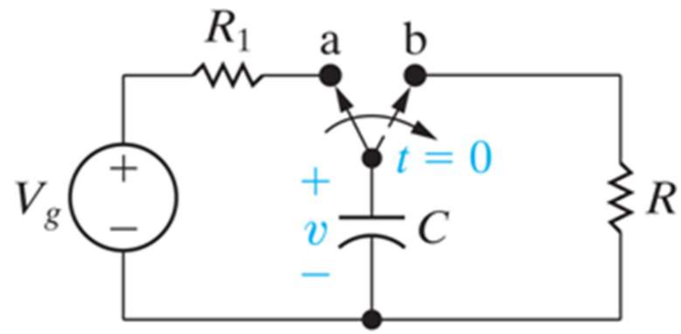


The problem: Find *v(t)* for *t* ≥ *0*. Note that the circuit to the left of the capacitor establishes an initial voltage drop *v(0)* across *C* at *t* = *0*.



Here is the circuit for the RC natural response problem. For $t < 0$, the capacitor can be replaced by

A. An open circuit
B. A short circuit
C. A constant current

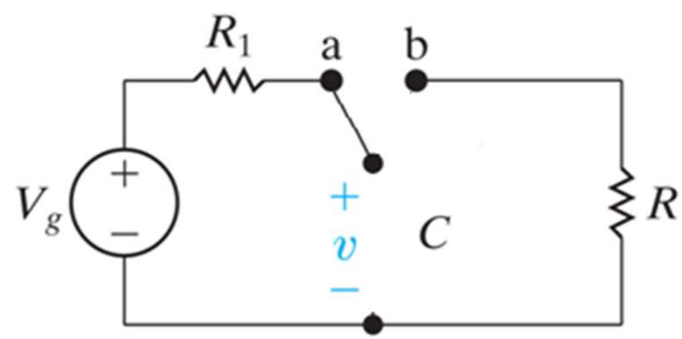


To evaluate the voltage drop across the capacitor for $t < 0$, put the switch in the "a" position and replace the capacitor with an open circuit. The voltage drop across the open circuit, positive at the top, is

- ✓ A. V_g
- ✗ B. $-V_g$
- ✗ C. $RV_g / (R + R_1)$

RC Natural Response

$t < 0:$



There is an open circuit here, so the current in this circuit is 0, and the voltage drop across R_1 is 0. Thus,

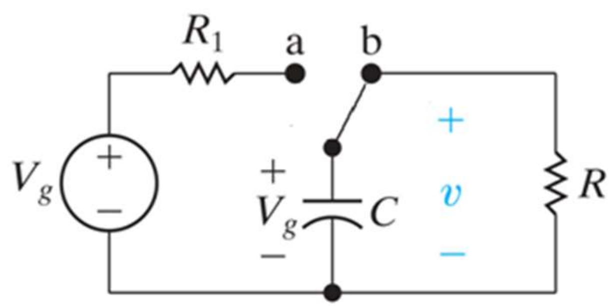
$$v = V_g, \qquad t < 0$$

Remember the continuity requirement for capacitors – the voltage drop across the capacitor must be continuous everywhere, so that the current through the capacitor remains finite. Thus,

$$v(0) = V_g$$

RC Natural Response

$t \geq 0:$



KCL: $$C\frac{dv(t)}{dt} + \frac{v(t)}{R} = 0$$

Let $v(t) = e^{-at}$ and $\dfrac{dv(t)}{dt} = -ae^{-at}$

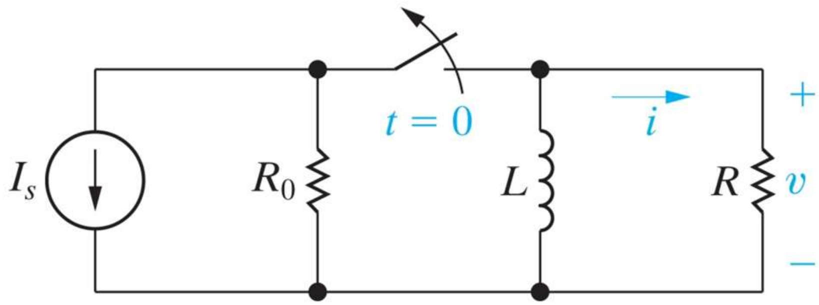
Substituting, $C(-ae^{-at}) + e^{-at}/R = 0$

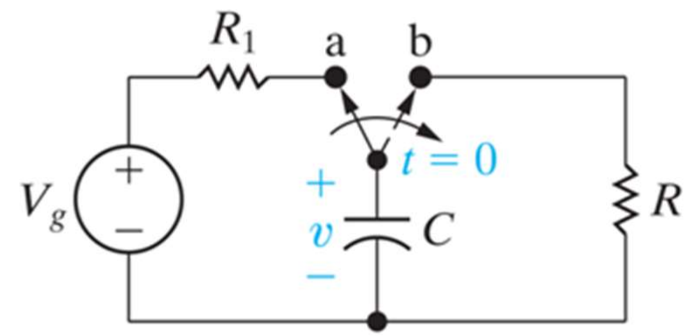
Rearranging, $(1/R - aC)e^{-at} = 0$ so $1/R - aC = 0$

Solving, $a = 1/RC$

But $v(0) = V_g$ so $v(t) = V_g e^{-(1/RC)t}, \quad t \geq 0$

RL/RC Natural Response





$$\text{KVL}: \quad L\frac{di(t)}{dt} + Ri(t) = 0$$

$$i(0) = I_s$$

$$i(t) = I_s e^{-(R/L)t}, \quad t \geq 0$$

$$\text{KCL}: \quad C\frac{dv(t)}{dt} + \frac{v(t)}{R} = 0$$

$$v(0) = V_g$$

$$v(t) = V_g e^{-(1/RC)t}, \quad t \geq 0$$

The form of the natural response is the same:

$$\text{IC}e^{-t/\tau}$$

IC is the initial condition and τ is the **time constant,** a measure of how quickly or slowly the exponential decays.

The form of the RL/RC natural response is

$$ICe^{-t/\tau}$$

The units for the time constant, τ, must be

A. s (seconds)
B. s^{-1} (inverse seconds)
C. τ has no units

The RL natural response is

$$i(t) = I_s e^{-(R/L)t}, \qquad t \geq 0$$

Therefore, the time constant, τ, for the RL circuit is

A. R/L
B. L/R*t*
C. L/R

RL/RC Natural Response

A general solution method:

1. Identify the variable of interest (hint – it's the variable that must be continuous in the circuit):
   - For RL, *i(t)* through L
   - For RC, *v(t)* across C
2. Find the initial value of this variable, either $i(0) = I_o$ or $v(0) = V_o$
   - From the problem statement
   - By analyzing the circuit for $t < 0$, with L replaced by a short circuit or C replaced by an open circuit
3. Find the time constant, τ
   - $\tau_{RL} = L/R_{eq}$ or $\tau_{RC} = R_{eq}C$
   - Note that R_{eq} is the equivalent resistance as seen from the inductor or capacitor for $t \geq 0$

RL/RC Natural Response

A general solution method, continued:

4. Write the expression for the variable of interest:

   $$x(t) = X_o e^{-t/\tau}, \quad t \geq 0$$

5. Use simple circuit analysis to calculate any other requested variables.