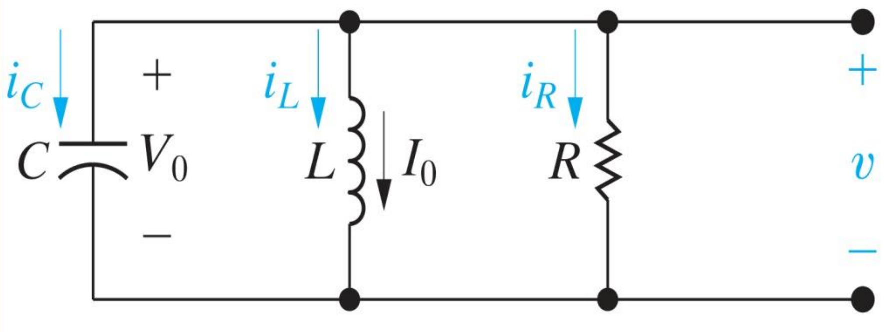
Natural and Step Response of Series & Parallel RLC Circuits (Second-order Circuits)

Objectives:

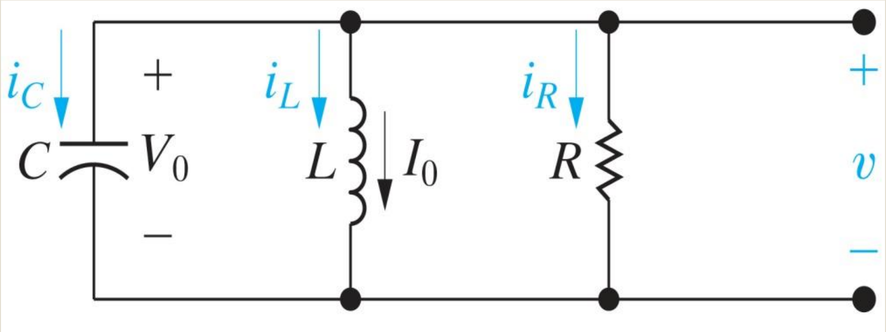
- ✓Determine the response form of the circuit
- ✓Natural response parallel RLC circuits
- ✓Natural response series RLC circuits
- ✓Step response of parallel and series RLC circuits

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for $t \geq 0$.



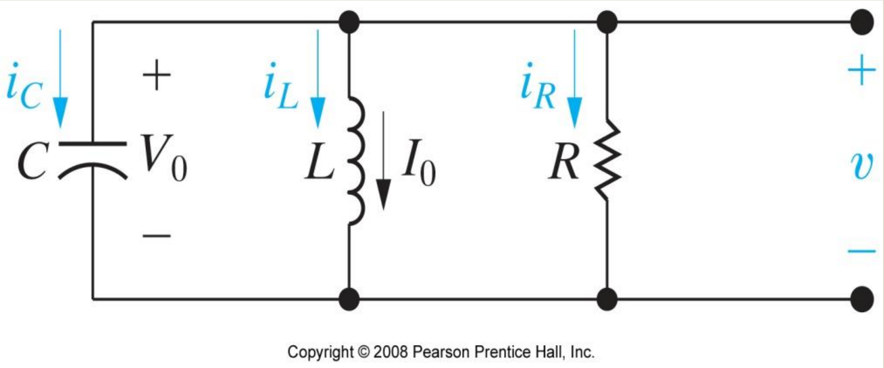
It is convenient to calculate *v(t)* for this circuit because



- ✗ A. The voltage must be continuous for all time
- ✗ B. The voltage is the same for all three components
- ✗ C. Once we have the voltage, it is pretty easy to calculate the branch current
- ✓ D. All of the above

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.



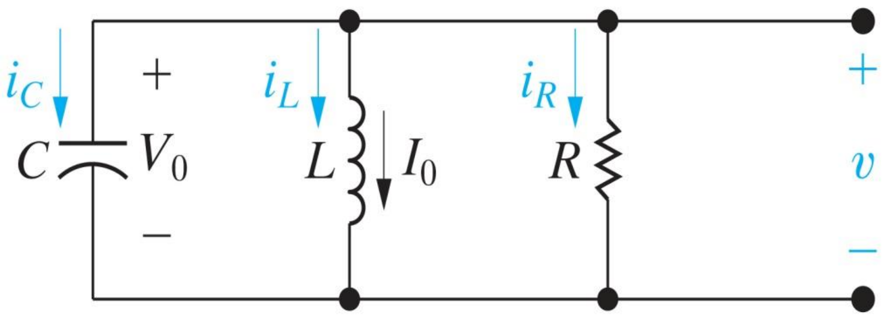
KCL : $$C\frac{dv(t)}{dt}+\frac{1}{L}\int_0^t v(x)dx+I_0+\frac{v(t)}{R}=0$$

Differentiate both sides to remove the integral : $$C\frac{d^2v(t)}{dt^2}+\frac{1}{L}v(t)+\frac{1}{R}\frac{dv(t)}{dt}=0$$

Divide both sides by C to place in standard form : $$\frac{d^2v(t)}{dt^2}+\frac{1}{LC}v(t)+\frac{1}{RC}\frac{dv(t)}{dt}=0$$

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for $t \geq 0$.



Describing equation: $$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) + \frac{1}{RC}\frac{dv(t)}{dt} = 0$$

This equation is

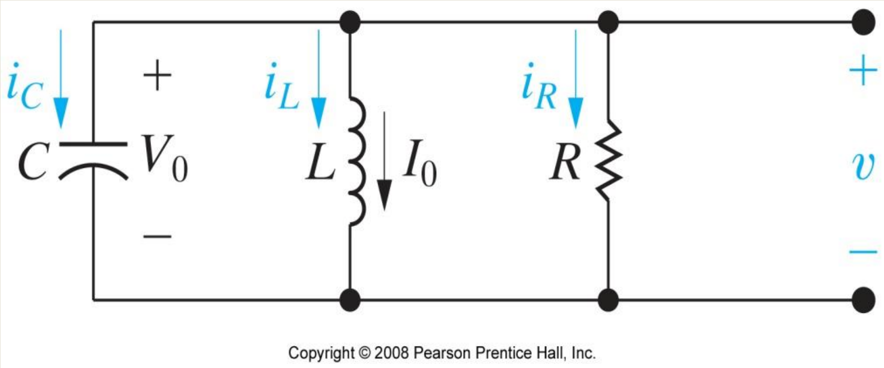
- ✓Second order
- ✓Homogeneous
- ✓Ordinary differential equation
- ✓With constant coefficients

Once again we want to pick a possible solution to this differential equation. This must be a function whose first AND second derivatives have the same form as the original function, so a possible candidate is

A. $K\sin \omega t$
B. Ke^{-at}
C. Kt^2

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.



Describing equation: $$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) + \frac{1}{RC}\frac{dv(t)}{dt} = 0$$

The circuit has two initial conditions that must be satisfied, so the solution for v(t) must have two constants. Use

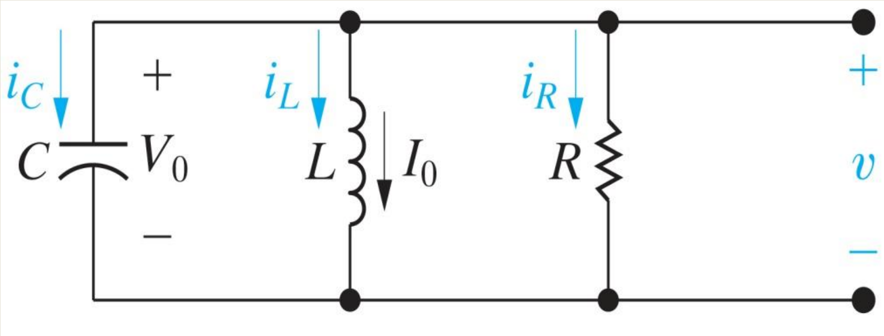
$v(t) = A_1e^{s_1t} + A_2e^{s_2t}$ V; Substitute:

$$(s_1^2A_1e^{s_1t} + s_2^2A_2e^{s_2t}) + \frac{1}{RC}(s_1A_1e^{s_1t} + s_2A_2e^{s_2t}) + \frac{1}{LC}(A_1e^{s_1t} + A_2e^{s_2t}) = 0$$

$$\Rightarrow \quad [s_1^2 + (1/RC)s_1 + (1/LC)]A_1e^{s_1t} + [s_2^2 + (1/RC)s_2 + (1/LC)]A_2e^{s_2t} = 0$$

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.



Describing equation: $$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) + \frac{1}{RC}\frac{dv(t)}{dt} = 0$$

Solution: $$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where s_1 and s_2 are solutions for the CHARACTERISTIC EQUATION:

$$s^2 + (1/RC)s + (1/LC) = 0$$

$$s^2 + (1/RC)s + (1/LC) = 0$$

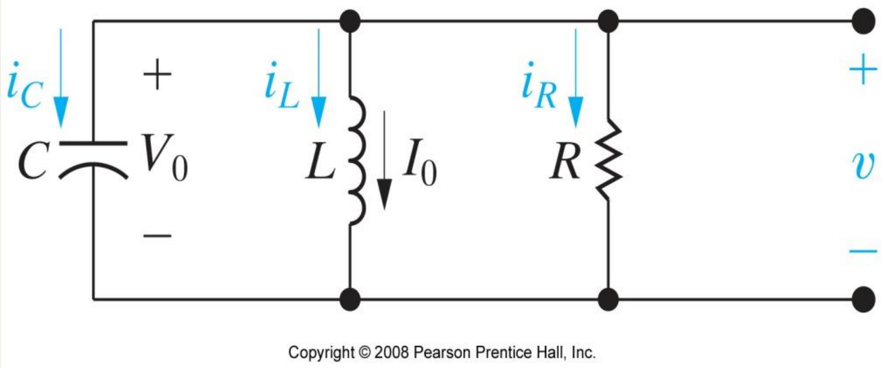
is called the "characteristic equation" because it characterizes the circuit.

✓ A. True

✗ B. False

Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.



The two solutions to the characteristic equation can be calculated using the quadratic formula:

$$s^2 + (1/RC)s + (1/LC) = 0; \qquad s_{1,2} = \frac{-(1/RC) \pm \sqrt{(1/RC)^2 - 4(1/LC)}}{2}$$

$$s_{1,2} = -(1/2RC) \pm \sqrt{(1/2RC)^2 - (1/LC)} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = \dfrac{1}{2RC}$ (the neper frequency in rad/s)

and $\omega_0 = \sqrt{\dfrac{1}{LC}}$ (the resonant radian frequency in rad/s)

So far, we know that the parallel RLC natural response is given by

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad \text{where} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

There are three different forms for s_1 and s_2. For a parallel RLC circuit with specific values of R, L and C, the form for s_1 and s_2 depends on

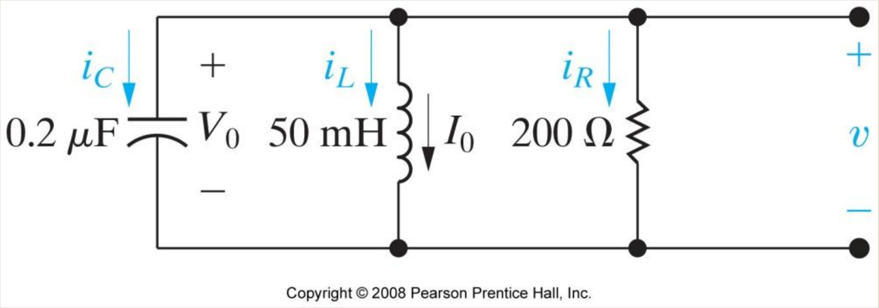
A. The value of α

B. The value of ω_o

C. The value of $(\alpha^2 - \omega_o^2)$

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for *t ≥ 0*.



$$\alpha = \frac{1}{2RC} = \frac{1}{2(200)(0.2\mu)} = 12{,}500 \text{ rad/s}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.05)(0.2\mu)}} = 10{,}000 \text{ rad/s}$$

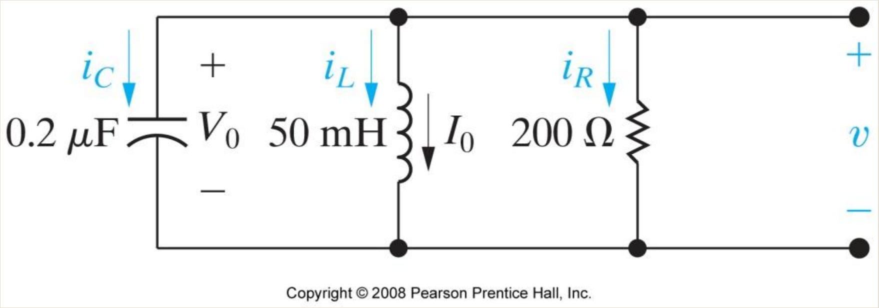
$\alpha^2 > \omega_o^2$ so this is the overdamped case!

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -12{,}500 \pm \sqrt{(12{,}500)^2 - (10{,}000)^2}$$

$$= -12{,}000 \pm 7500 \qquad \Rightarrow \qquad s_1 = -5000 \text{ rad/s}, s_2 = -20{,}000 \text{ rad/s}$$

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for $t \geq 0$.



$$v(t) = A_1 e^{-5000t} + A_2 e^{-20,000t} \text{ V, for } t \geq 0$$

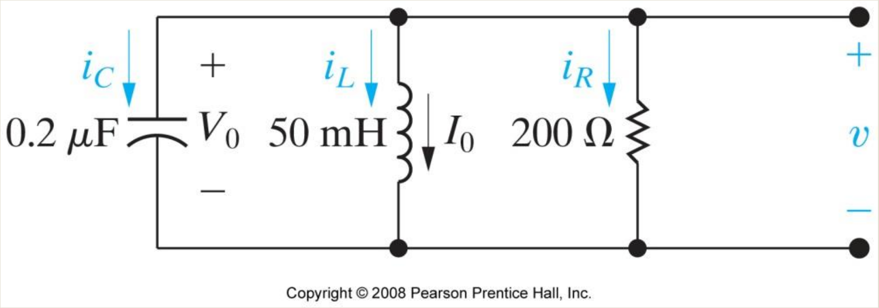
Now we must use the coefficients in the equation to satisfy the initial conditions in the circuit:

$$v(t)\big|_{t=0} \text{ in the equation} = v(t)\big|_{t=0} \text{ in the circuit}$$

$$\left.\frac{dv(t)}{dt}\right|_{t=0} \text{ in the equation} = \left.\frac{dv(t)}{dt}\right|_{t=0} \text{ in the circuit}$$

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for *t* ≥ *0*.



Equation: $$v(0) = A_1 e^{-5000(0)} + A_2 e^{-20,000(0)} = A_1 + A_2$$

Circuit: $$v(0) = V_0 = 12\,\text{V}$$

$$\Rightarrow \quad A_1 + A_2 = 12$$

Equation: $$\frac{dv(0)}{dt} = (-5000)A_1 e^{-5000(0)} + (-20,000)A_2 e^{-20,000(0)}$$

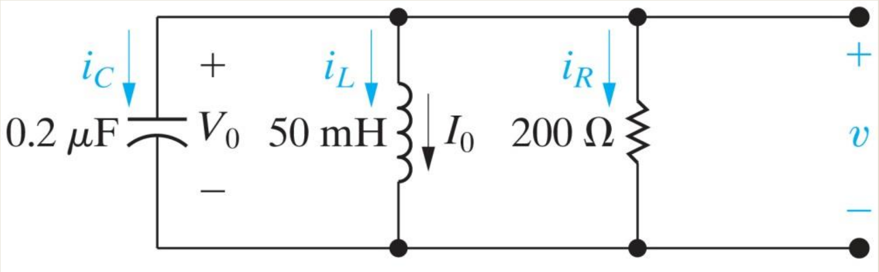
$$= -5000A_1 - 20,000A_2$$

Now we need the initial value of the first derivative of the voltage from the circuit. The describing equation of which circuit component involves *dv(t)/dt*?

A. The resistor
B. The inductor
C. The capacitor

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for *t ≥ 0*.



Equation:

$$\frac{dv(0)}{dt} = (-5000)A_1e^{-5000(0)} + (-20,000)A_2e^{-20,000(0)}$$

$$= -5000A_1 - 20,000A_2$$

Circuit:

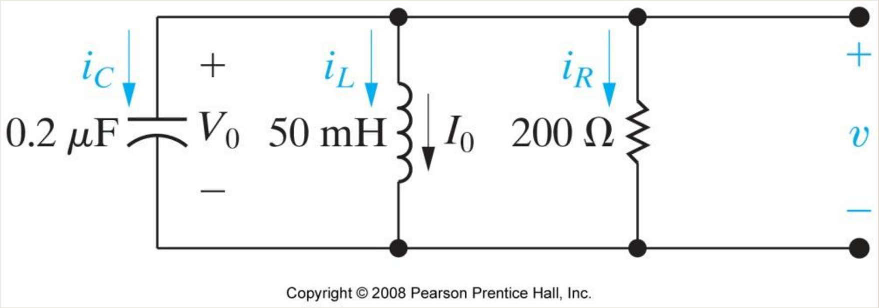
$$i_C(t) = C\frac{dv(t)}{dt} \Rightarrow \frac{dv(0)}{dt} = \frac{1}{C}i_C(0) = \frac{1}{C}(-i_L(0) - i_R(0))$$

$$\frac{dv(0)}{dt} = \frac{1}{C}\left(-i_L(0) - \frac{v(0)}{R}\right) = \frac{1}{0.2\mu}\left(-0.03 - \frac{12}{200}\right) = -450,000\text{ V/s}$$

$$\Rightarrow -5000A_1 - 20,000A_2 = -450,000$$

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for *t* ≥ *0*.



$$v(t) = A_1 e^{-5000t} + A_2 e^{-20,000t} \text{ V, for } t \geq 0$$

$$A_1 + A_2 = 12; \qquad -5000A_1 - 20,000A_2 = -450,000$$

Solving simultaneously, $A_1 = -14, \qquad A_2 = 26$

Thus,

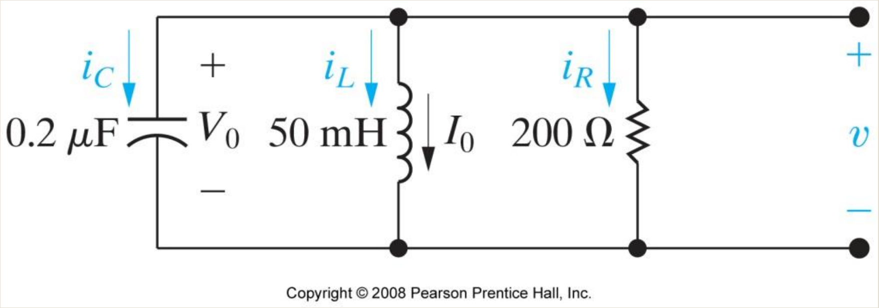
$$v(t) = -14e^{-5000t} + 26e^{-20,000t} \text{ V, for } t \geq 0$$

Checks: $v(0) = -14 + 26 = 12\text{ V}$ (OK)

$v(\infty) = 0$ (OK)

Natural Response – Overdamped Example

Given V_0 = 12 V and I_0 = 30 mA, find *v(t)* for $t \geq 0$.



You can solve this problem using the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate α and ω_0.
4. Compare the values of α and ω_o to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for *v(t)*, $t \geq 0$.
7. Solve for any other quantities requested in the problem.