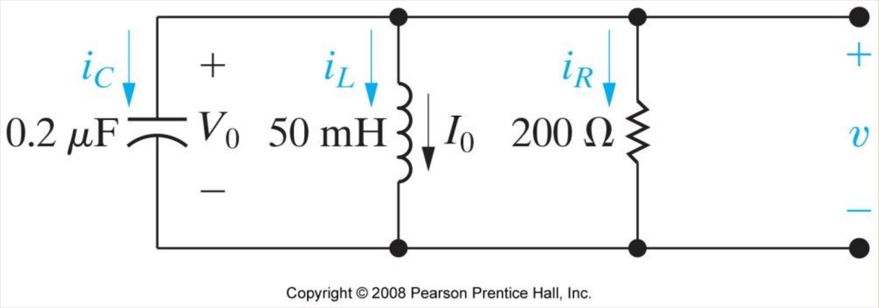
# Natural Response – Overdamped Example

Given $V_0$ = 12 V and $I_0$ = 30 mA, find *v(t)* for $t \geq 0$.




You can solve this problem using the Second-Order Circuits table:

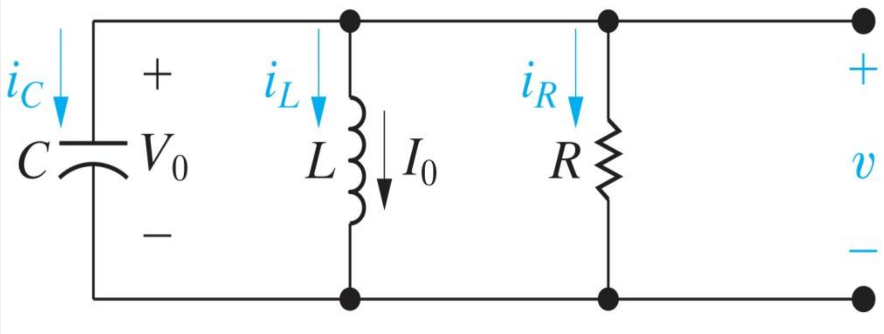
1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
4. Compare the values of $\alpha$ and $\omega_o$ to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for *v(t)*, $t \geq 0$.
7. Solve for any other quantities requested in the problem.

The values of the ___________ determine whether the response is overdamped, underdamped, or critically damped

A. Initial conditions
B. R, L, and C components
C. Independent sources

# Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.





Recap:

$$s^2 + (1/RC)s + (1/LC) = 0; \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \qquad \alpha = \frac{1}{2RC}; \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

$\alpha^2 > \omega_0^2:$ overdamped, so $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$\alpha^2 < \omega_0^2:$ underdamped, so $s_{1,2} = -\alpha \pm j\omega_d$ where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

Note Euler's identity: $e^{jx} = \cos x + j\sin x; \qquad e^{-jx} = \cos x - j\sin x$

$$v(t) = A_1 e^{-\alpha t}(\cos\omega_d t + j\sin\omega_d t) + A_2 e^{-\alpha t}(\cos\omega_d t - j\sin\omega_d t)$$

$$= e^{-\alpha t}\cos\omega_d t(A_1 + A_2) + e^{-\alpha t}\sin\omega_d t(jA_1 - jA_2)$$

$$\Rightarrow \qquad v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

When the response is underdamped, the voltage is given by the equation

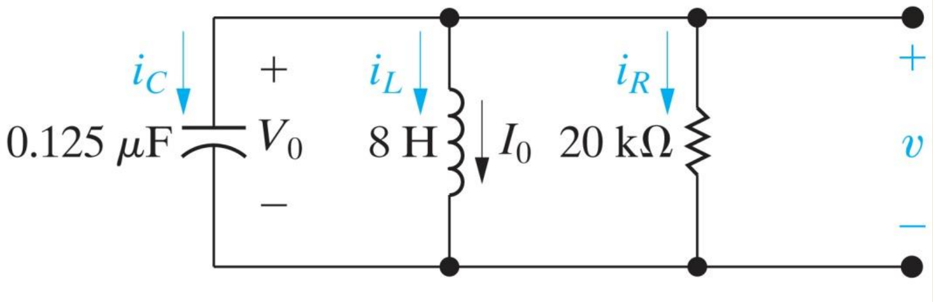
$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

In this equation, the coefficients $B_1$ and $B_2$ are

A. Real numbers
B. Imaginary numbers
C. Complex conjugate numbers

# Natural Response – Underdamped Example

Given $V_0$ = 0 V and $I_0$ = –12.25 mA, find *v(t)* for *t ≥ 0*.





$$\alpha = \frac{1}{2RC} = \frac{1}{2(20{,}000)(0.125\mu)} = 200 \text{ rad/s}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(8)(0.125\mu)}} = 1000 \text{ rad/s}$$

$\alpha^2 < \omega_o^2$ so this is the underdamped case!

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(1000)^2 - (200)^2} = 979.8 \text{ rad/s}$$

$$\Rightarrow \qquad v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$= B_1 e^{-200t}\cos 979.8t + B_2 e^{-200t}\sin 979.8t \text{ V}, t \geq 0$$

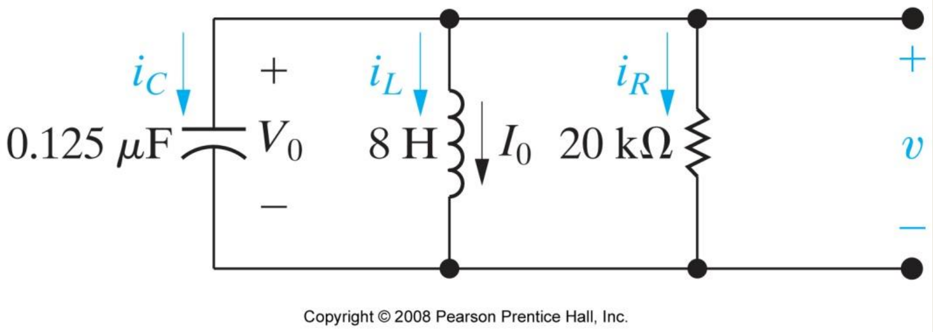
Now we evaluate v(0) and dv(0)/dt from the equation for v(t), and set those values equal to v(0) and dv(0)/dt from the circuit, solving for $B_1$ and $B_2$. The values for v(0) and dv(0)/dt from the circuit do not depend on whether the response is overdamped, underdamped, or critically damped.

✓ A. True

✗ B. False

# Natural Response – Underdamped Example

Given $V_0$ = 0 V and $I_0$ = –12.25 mA, find *v(t)* for *t ≥ 0*.



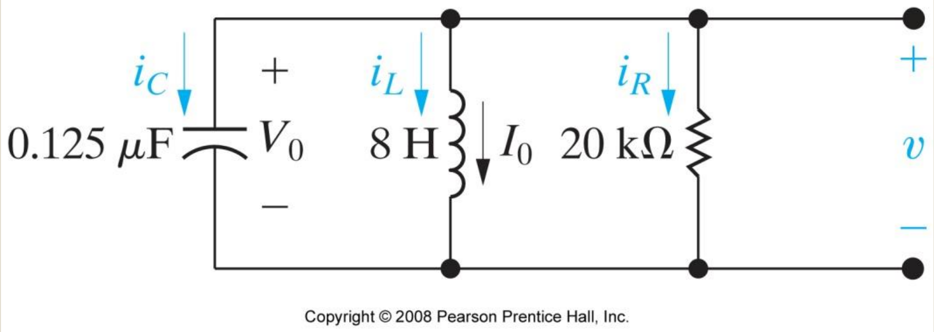

Equation: $v(0) = B_1 e^{-200(0)} \cos 979.8(0) + B_2 e^{-200(0)} \sin 979.8(0) = B_1$

Circuit: $v(0) = V_0 = 0\,\text{V}$

$\Rightarrow \quad B_1 = 0$

# Natural Response – Underdamped Example

Given $V_0$ = 0 V and $I_0$ = –12.25 mA, find *v(t)* for *t ≥ 0*.




$$\text{Equation:}\quad \frac{dv(0)}{dt} = (-200)B_1e^{-200(0)}\cos 979.8(0) - 979.8B_1e^{-200(0)}\sin 979.8(0)$$

$$+(-200)B_2e^{-200(0)}\sin 979.8(0) + 979.8B_2e^{-200(0)}\cos 979.8(0)$$

$$= -\alpha B_1 + \omega_d B_2 = -200B_1 + 979.8B_2$$

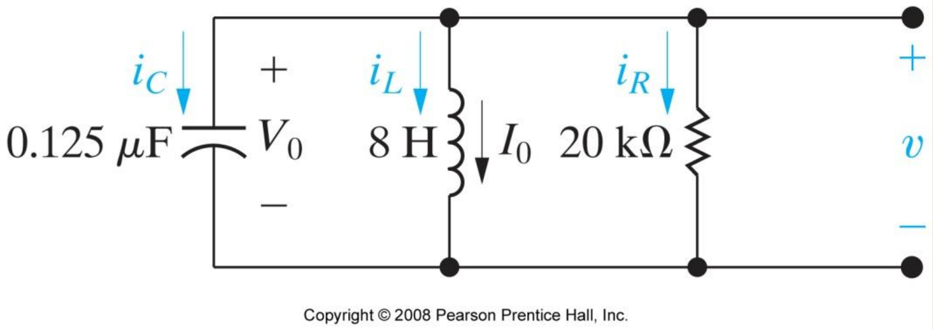
$$\text{Circuit:}\quad \frac{dv_C(0)}{dt} = \frac{1}{C}i_C(0) = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

$$= \frac{1}{0.125\mu}\left(-(-0.01225) - \frac{0}{20{,}000}\right) = 98{,}000\,\text{V/s}$$

$$\Rightarrow\quad -200B_1 + 979.8B_2 = 98{,}000\,\text{V/s}$$

# Natural Response – Underdamped Example

Given $V_0 = 0$ V and $I_0 = -12.25$ mA, find *v(t)* for *t ≥ 0*.




$$v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$v(0) = B_1 = V_0 = 0;$$

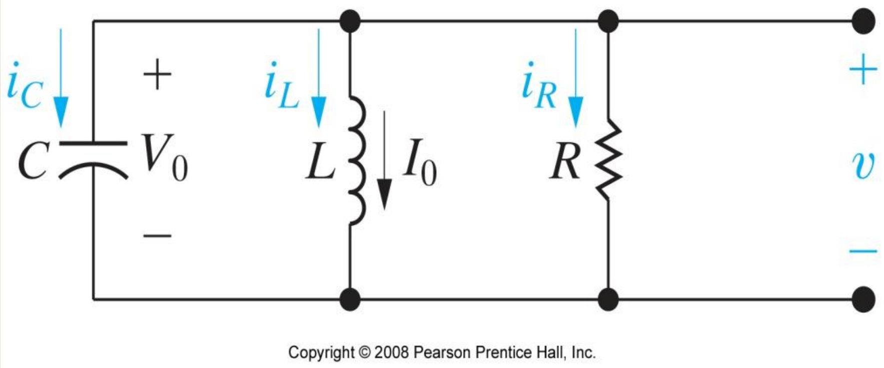
$$\frac{dv(0)}{dt} = -\alpha B_1 + \omega_d B_2 = -200B_1 + 979.8B_2$$

$$= \frac{1}{C}\left(-I_0 - V_0/R\right) = 98{,}000 \qquad \Rightarrow \qquad B_2 = 100$$

$$\therefore \qquad v(t) = 100e^{-200t}\sin 979.8t\ \text{V}, t \geq 0$$

# Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.




Recap:

$$s^2 + (1/RC)s + (1/LC) = 0; \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \qquad \alpha = \frac{1}{2RC}; \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

$\alpha^2 > \omega_0^2$ : overdamped, so $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$\alpha^2 < \omega_0^2$ : underdamped, so $v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$\alpha^2 = \omega_0^2$ : Critically damped, so $s_{1,2} = -\alpha \pm 0 = -\alpha$

When the response is critically damped, a reasonable expression for the voltage is

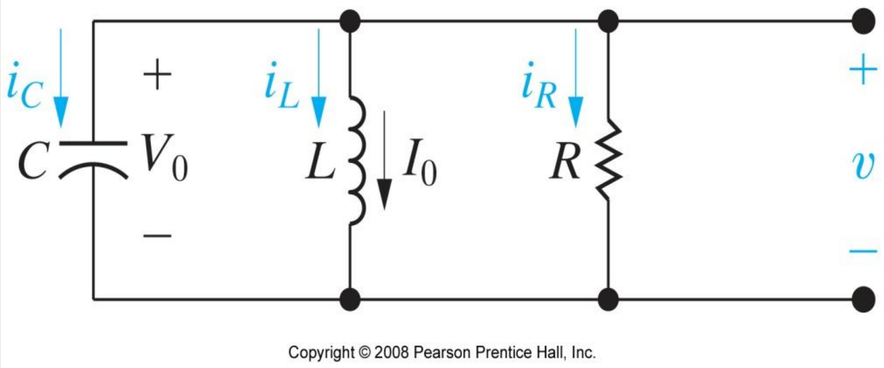
$$v(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} \text{ V}, t \geq 0$$

A. True

B. False

# Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for *t ≥ 0*.



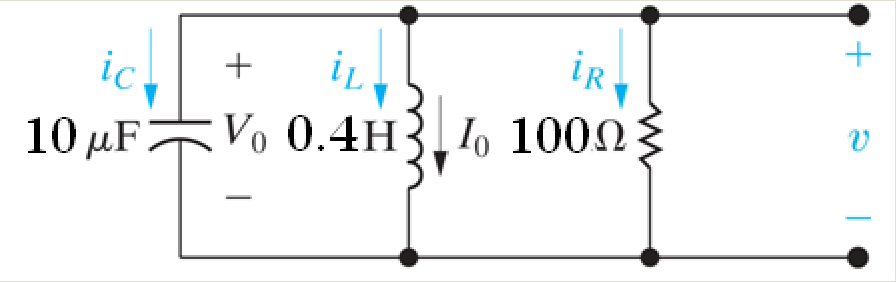

When the circuit's response is critically damped, the assumed form of the solution we have been using up until now does not provide enough unknown coefficients to satisfy the two initial conditions from the circuit. Therefore, we use a different solution form:

$\alpha^2 = \omega_0^2$ : Critically damped so

$$v(t) = D_1 t e^{s_1 t} + D_2 e^{s_2 t} = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

# Natural Response – Critically damped Example

Given $V_0$ = 50 V and $I_0$ = 250 mA, find *v(t)* for *t ≥ 0*.




$$\alpha = \frac{1}{2RC} = \frac{1}{2(100)(10\mu)} = 500\text{rad/s}$$

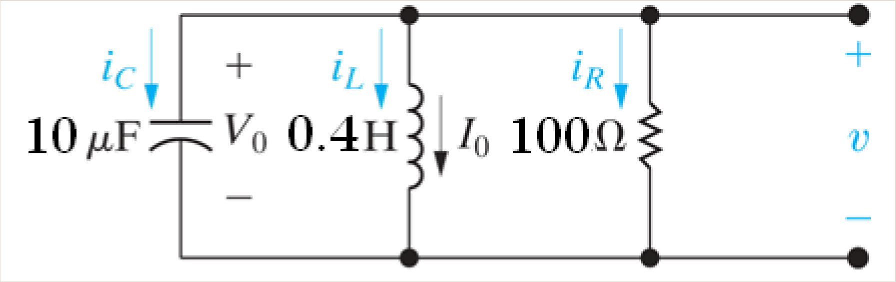
$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.4)(10\mu)}} = 500\text{rad/s}$$

$\alpha^2 = \omega_o^2$ so this is the critically damped case!

$$\Rightarrow \qquad v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} = D_1 t e^{-500t} + D_2 e^{-500t} \text{ V}, t \geq 0$$

# Natural Response – Critically damped Example

Given $V_0$ = 50 V and $I_0$ = 250 mA, find *v(t)* for *t ≥ 0*.



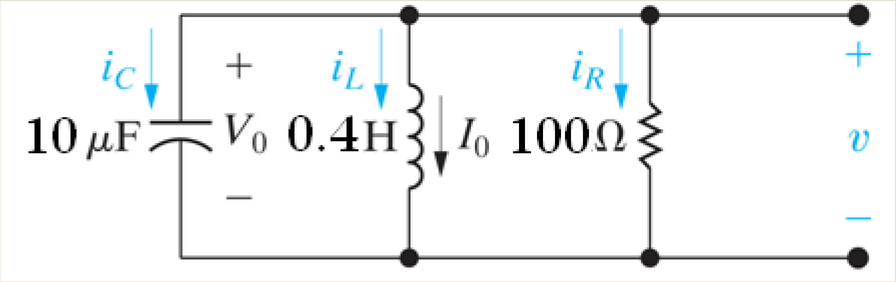

Use the initial conditions from the equation and from the circuit to solve for the unknown coefficients.

Equation: $v(0) = D_1(0)e^{-500(0)} + D_2 e^{-500(0)} = D_2$

Circuit: $v(0) = V_0 = 50\text{V} \quad \Rightarrow \quad D_2 = 50$

# Natural Response – Critically damped Example

Given $V_0$ = 50 V and $I_0$ = 250 mA, find *v(t)* for *t ≥ 0*.




$$\text{Equation:} \quad \frac{dv(0)}{dt} = D_1 e^{-500(0)} + D_1(-500)(0)e^{-500(0)} + D_2(-500)e^{-500(0)}$$

$$= D_1 - 500D_2$$

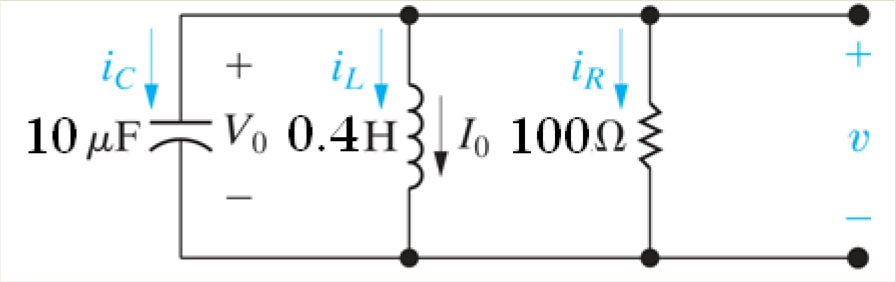
$$\text{Circuit:} \quad \frac{dv_C(0)}{dt} = \frac{1}{C} i_C(0) = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

$$= \frac{1}{10\mu}\left(-0.25 - \frac{50}{100}\right) = -75{,}000\,\text{V/s}$$

$$\Rightarrow \quad D_1 - 500D_2 = -75{,}000\,\text{V/s}$$

# Natural Response – Critically damped Example

Given $V_0 = 50$ V and $I_0 = 250$ mA, find *v(t)* for *t ≥ 0*.




$$v(t) = D_1 t e^{-500t} + D_2 e^{-500t}$$

$$v(0) = D_2 = V_0 = 50;$$

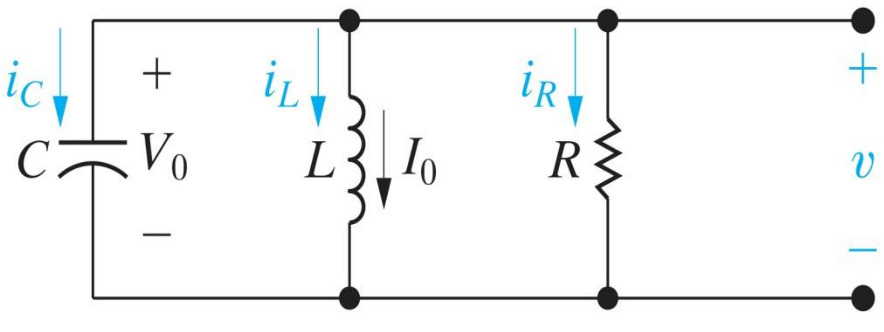
$$\frac{dv(0)}{dt} = D_1 - \alpha D_2 = D_1 - 500 D_2$$

$$= \frac{1}{C}\left(-I_0 - V_0/R\right) = -78{,}000 \qquad \Rightarrow \qquad D_1 = -50{,}000$$

$$\therefore \qquad v(t) = -50{,}000 t e^{-500t} + 50 e^{-500t} \text{ V}, t \geq 0$$

# Natural Response of Parallel RLC Circuits – Summary

The problem – given initial energy stored in the inductor and/or capacitor, find *v(t)* for $t \geq 0$.





Use the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
4. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for *v(t)*, $t \geq 0$.
7. Solve for any other quantities requested in the problem.